The Schurman Parabola - Part I

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In Economics and finance we often have the case where a revenue stream or asset base has a short-term growth rate that is unstainably high and decreases to a sustainable long-term growth rate over time. In other words, the equation for base value has a first derivative that is positive (the base is increasing over time) and a second derivative that is negative (the rate of increase declines over time). Our goal in this white paper is to develop an integrable equation for this type of scenario. To this end we will use the following hypothetical problem...

Table 1 - Hypothetical Problem

Description	Parameter	Value
Annualized revenue at time zero	B_0	\$1,000,000
Short-term annual growth rate	r_0	0.50
Long-term annual growth rate	r_T	0.04
Transition period (in years)	T	7.00

In this problem we have a revenue stream where the short-term unsustainable growth rate of 50% decreases to the long-term sustainable growth rate of 4% over a time period that is seven years in length. Our goal is to develop a closed-form equation that answers the following question...

Question: What is total revenue received over the seven year period?

The Base Value Curve

We will define the variable B_t to be our base value at time t and the variable r_t to be our mean-reverting growth rate at time t. We will define the change in base value over time to be in accordance with the following ordinary differential equation...

$$\delta B_t = r_t B_t \delta t \text{ ...such that... } r_t = \frac{\delta B_t}{B_t} \times \frac{1}{\delta t} \tag{1}$$

We will use a parabola to model base value at time $0 \leq t \leq T$. Note that a parabola has the following functional form...

$$f(x) = ax^2 + bx + c \tag{2}$$

Using Equation (2) above as our guide our equation for base value at time t is...

$$B_t = a\,t^2 + b\,t + c \tag{3}$$

We will need the derivative of Equation (3) above with respect to time, which is...

$$\frac{\delta B_t}{\delta t} = 2\,a\,t + b \text{ ...such that... } \delta B_t = (2\,a\,t + b)\,\delta t \tag{4}$$

Note that the parabola will be increasing as long as Equation (4) above is positive but because we will only use a portion of the parabola (i.e. over the time interval $[0, T]$) this should not be a problem. This statement in equation form is...

$$\text{Parabola is decreasing when... } t > -\frac{b}{2a} \text{ ...but... } T < t \tag{5}$$

We will also need an equation for the area underneath the parabola over the time interval $[0, t]$. Using Equation (3) above the equation for the area underneath the parabola is...

$$\int_0^t B_u \, \delta u = \frac{1}{3} \, a \, u^3 + \frac{1}{2} \, b \, u^2 + c \, u \Big[_0^t = \frac{1}{3} \, a \, t^3 + \frac{1}{2} \, b \, t^2 + c \, t \tag{6}$$

Calibrating Our Parabola

To solve the hypothetical problem above we will define base value to be annualized revenue at time t. The equation for total revenue received over the time interval $[0, t]$, and the answer to our hypothetical problem, is therefore...

$$\text{Total revenue received} = \int_0^t B_u \, \delta u \tag{7}$$

Since we know base value at time zero (B_0) we will first solve for parabola parameter $[c]$. Using Equation (3) above and the parameters in **Table 1** the value of parabola parameter c is...

$$B_0 = a \, (0)^2 + b \, (0) + c \;\; \text{...such that...} \;\; c = B_0 \tag{8}$$

Since we know the short-term growth rate (r_0), the long-term growth rate (r_T) and the transition period (T) we can solve for parabola parameters $[a]$ and $[b]$. Using Equations (1), (3) and (4) above the equation for the periodic growth rate at time t is...

$$r_t = \frac{\delta B_t}{B_t} \times \frac{1}{\delta t} = \frac{(2 \, a \, t + b) \, \delta t}{a \, t^2 + b \, t + c} \times \frac{1}{\delta t} = \frac{2 \, a \, t + b}{a \, t^2 + b \, t + c} \tag{9}$$

We will use the short-term growth rate to solve for parabola parameter $[b]$. Using Equation (9) above and the parameters in **Table 1** the value of parabola parameter b is...

$$r_0 = \frac{2 \, a(0) + b}{a \, (0)^2 + b \, (0) + c} = \frac{b}{c} \;\; \text{...such that...} \;\; b = r_0 \, c \tag{10}$$

Now that we have values for parabola parameters b and c we will use the long-term rate to solve for parabola parameter $[a]$. Using Equation (9) above and the parameters in **Table 1** the value of parabola parameter a is...

$$r_T = \frac{2 \, a \, (T) + b}{a \, (T)^2 + b \, (T) + c} = \frac{2 \, a \, T + b}{a \, T^2 + b \, T + c} \;\; \text{...such that...} \;\; a = \frac{r_T \, (bT + c) - b}{2T - r_T \, T^2} \tag{11}$$

The Answer To Our Hypothetical Problem

To solve our hypothetical problem we need to calculate total revenue received over the time interval $[0, 7]$ using the parameters in Table 1 above. To accomplish this goal we need to calculate the area underneath a parabola. To do this we will need to determine the values of parabola parameters a, b and c.

Our first step will be to solve for parabola parameter **c**. Using Equation (8) above and the data in Table 1 the value of parabola parameter c is...

$$c = B_0 = 1000 \tag{12}$$

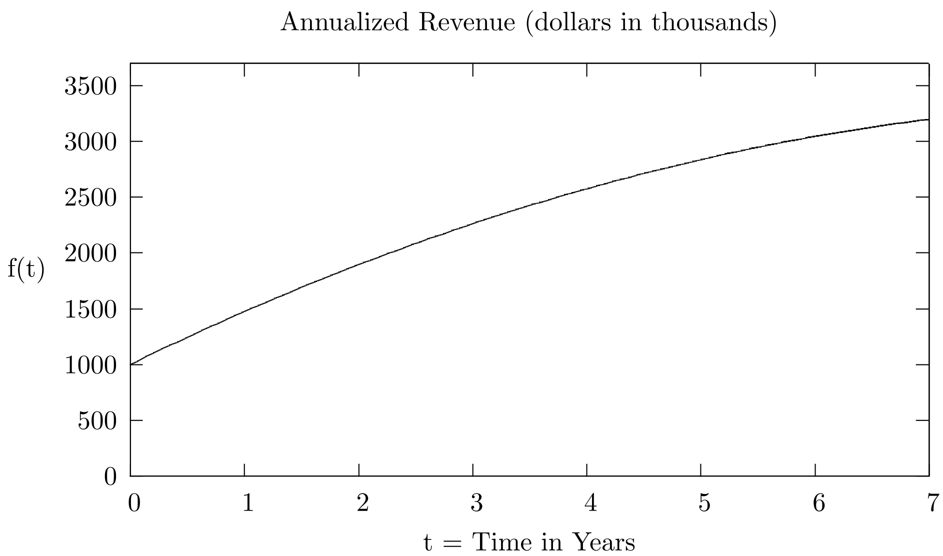
Our next step will be to solve for parabola parameter **b**. Using Equations (10) and (12) above and the data in Table 1 the value of parabola parameter b is...

$$b = r_0 \, c = 0.50 \times 1000 = 500 \tag{13}$$

Our next step will be to solve for parabola parameter **a**. Using Equations (11), (12) and (13) above and the data in Table 1 the value of parabola parameter a is...

$$a = \frac{r_T \, (bT + c) - b}{2T - r_T \, T^2} = \frac{0.04 \times (500 \times 7 + 1000) - 500}{2 \times 7 - 0.04 \times 7^2} = -26.5781 \tag{14}$$

Using parabola Equation (3) and the parabola parameter estimates via Equations (12), (13) and (14) above the graph of our annualized revenue equation looks like this...



Answer: Using Equations (6), (12), (13) and (14) above and the parameters in Table 1 the answer to our hypothetical problem is...

$$\text{Total revenue received} = \frac{1}{3}\,(-26.5781)\,(7)^3 + \frac{1}{2}\,(500)\,(7)^2 + (1000)(7) = 16211 \tag{15}$$

Total revenue received over the seven year period, and the answer to our hypothetical problem, is approximately \$16,211.